CALCULUS BC
WORKSHEET 1 ON TAYLOR POLYNOMIALS

Work the following on **notebook paper**. Use your calculator only on problem 8(b) , 9(b), and 10(a). Show all work.

1. Find a fourth-degree Maclaurin polynomial for $f(x) = e^{3x}$.
2. Find a sixth-degree Maclaurin polynomial for $f(x) = \cos x$.
3. Find a fifth-degree Maclaurin polynomial for $f(x) = \dfrac{1}{x+1}$.
4. Find a third-degree Taylor polynomial for $f(x) = \sin x$, centered at $x = \dfrac{\pi}{6}$
5. Find a fifth-degree Taylor polynomial for $f(x) = \dfrac{1}{1-x}$, centered at $x = 2$.
6. Find a third-degree Taylor polynomial for $f(x) = e^{(x-4)}$, centered at $x = 4$.
7. Find a fifth-degree Taylor polynomial for $f(x) = \ln(x-1)$, centered at $x = 2$.

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8. (a) Find a Taylor polynomial of degree $n = 4$ for $f(x) = e^{2x}$ centered at $x = 3$.
   (b) Find $P_4(3.31)$. What is the value of $f(3.31)$ and the value of $|f(3.31) - P_4(3.31)|$?

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9. Let $g$ be a function which has derivatives of all orders for all real numbers. Assume
   $g(5) = 3,\ g'(5) = -2,\ g''(5) = 7,\ g'''(5) = -3.$

(a) Write the Taylor polynomial of degree 3 for $g$ centered at $x = 5$.
(b) Use the polynomial that you found in part (a) to approximate $g(4.9)$.

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10. (1998 BC 3)
    Let $f$ be a function that has derivatives of all orders for all real numbers. Assume
    $f(0) = 5,\ f'(0) = -3,\ f''(0) = 1,\ f'''(0) = 4.$

(a) Write the third-degree Taylor polynomial for $f$ about $x = 0$, and use it to approximate $f(0.2)$.
(b) Write the fourth-degree Taylor polynomial for $g$, where $g(x) = f(x^2)$, about $x = 0$.
    (Hint: Substitute $x^2$ in place of $x$ in your answer to (a).)
(c) Write the third-degree Taylor polynomial for $h$, where $h(x) = \int_0^x f(t)\,dt$, about $x = 0$.
(d) Let $h$ be defined as in part (c). Given that $f(1) = 3$, either find the exact value of $h(1)$ or explain why it cannot be determined.

Answers to Worksheet 1 on Taylor Polynomials

1. $1+3x+\dfrac{9x^2}{2!}+\dfrac{27x^3}{3!}+\dfrac{81x^4}{4!}$

2. $1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\dfrac{x^6}{6!}$

3. $1-x+x^2-x^3+x^4-x^5$

4. $\dfrac{1}{2}+\dfrac{\sqrt{3}}{2}\left(x-\dfrac{\pi}{6}\right)-\dfrac{\left(x-\dfrac{\pi}{6}\right)^2}{2\cdot 2!}-\dfrac{\sqrt{3}\left(x-\dfrac{\pi}{6}\right)^3}{2\cdot 3!}$

5. $-1+(x-2)-(x-2)^2+(x-2)^3-(x-2)^4+(x-2)^5$

6. $1+(x-4)+\dfrac{(x-4)^2}{2!}+\dfrac{(x-4)^3}{3!}$

7. $(x-2)-\dfrac{(x-2)^2}{2}+\dfrac{(x-2)^3}{3}-\dfrac{(x-2)^4}{4}+\dfrac{(x-2)^5}{5}$

8. (a) $e^6+2e^6(x-3)+\dfrac{4e^6(x-3)^2}{2!}+\dfrac{8e^6(x-3)^3}{3!}+\dfrac{16e^6(x-3)^4}{4!}$

(b) $P_4(3.31)=749.602$

$f(3.31)=749.945$

$|f(3.31)-P_4(3.31)|=0.343$

9. (a) $3-2(x-5)+\dfrac{7(x-5)^2}{2!}-\dfrac{3(x-5)^3}{3!}$

(b) 3.236

10. (a) $5-3x+\dfrac{1}{2}x^2+\dfrac{2}{3}x^3$, $f(0.2)\approx 4.425$

(b) $5-3x^2+\dfrac{1}{2}x^4$

(c) $5x-\dfrac{3}{2}x^2+\dfrac{1}{6}x^3$

(d) $h(1)=\int_0^1 f(t)\,dt$. The exact value of $h(1)$ cannot be determined because $f(t)$ is known only for $t=0$ and $t=1$.

CALCULUS BC
WORKSHEET 2 ON TAYLOR POLYNOMIALS

Work the following on **notebook paper**. Use your calculator only on problem 9(b). Show all work.

1. Suppose the function $f(x)$ is approximated near $x = 5$ by a third-degree Taylor polynomial $P_3(x) = -3 + 7(x-5)^2 - 2(x-5)^3$. Give the value of:

(a) Give the value of: $f(5)$, $f'(5)$, $f''(5)$, and $f'''(5)$..

(b) Does $f$ have a local maximum, a local minimum, or neither at $x = 5$? Justify your answer.

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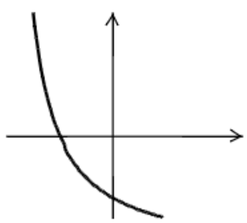
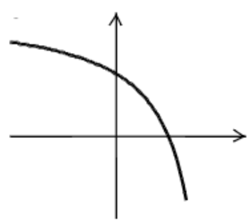
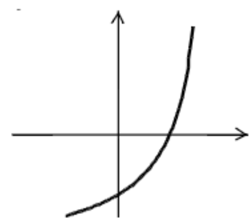
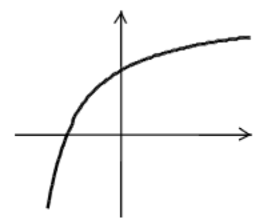
For problems 2 – 5, suppose that $P_2(x) = a + bx + cx^2$ is the second-degree Taylor polynomial for $f$ about $x = 0$. What are the signs of $a$, $b$, and $c$ if $f$ has the graph pictured on the below? Explain your reasoning.

2. 

3. 

4. 

5. 

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6. Find a fifth–degree Maclaurin polynomial for $f(x) = \sin(3x)$.

7. Find a fifth–degree Taylor polynomial for $f(x) = \ln(x-1)$ centered at $x = 2$.

8. Find a fourth–degree Taylor polynomial for $f(x) = e^{(x-4)}$ centered at $x = 4$.

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9. (Modified form of 2000 BC 3)
The Taylor series about $x = 5$ for a certain function $f$ converges to $f(x)$ for all $x$ in the interval of convergence. The nth derivative of $f$ at $x = 5$ is given by $f^{(n)}(5) = \dfrac{(-1)^n n!}{2^n (n+2)}$ and $f(5) = \dfrac{1}{2}$.

(a) Write the third-degree Taylor polynomial for $f$ about $x = 5$.

(b) Use your answer to (a) to approximate the value of $f(6)$.

Answers to Worksheet 2 on Taylor Polynomials

1. (a) $f(5) = -3$, $f'(5) = 0$, $f''(5) = 7 \cdot 2!$ or $14$, $f'''(5) = -2 \cdot 3!$ or $-12$

   (b) Since $f'(5) = 0$ and $f''(5)$ is positive, $f$ has a local minimum at $x = 5$ by the Second Derivative Test.

2. $f(0) = a$, the constant in the Taylor polynomial. Since the $y$-intercept is negative, $f(0) = -$. The graph of $f$ is decreasing so $f'(0) = +$. The graph of $f$ is concave up so $f''(0) = +$.

3. Since the $y$-intercept is positive, $f(0) = +$. The graph of $f$ is decreasing so $f'(0) = -$. The graph of $f$ is concave down so $f''(0) = -$.

4. Since the $y$-intercept is negative, $f(0) = -$. The graph of $f$ is increasing so $f'(0) = +$. The graph of $f$ is concave up so $f''(0) = +$.

5. Since the $y$-intercept is positive, $f(0) = +$. The graph of $f$ is increasing so $f'(0) = +$. The graph of $f$ is concave down so $f''(0) = -$.

6. $3x - \dfrac{27x^3}{3!} + \dfrac{243x^5}{5!}$

7. $(x-2) - \dfrac{(x-2)^2}{2} + \dfrac{(x-2)^3}{3} - \dfrac{(x-2)^4}{4} + \dfrac{(x-2)^5}{5}$

8. $1 + (x-4) + \dfrac{(x-4)^2}{2!} + \dfrac{(x-4)^3}{3!} + \dfrac{(x-4)^4}{4!}$

9. (a) $\dfrac{1}{2} - \dfrac{x-5}{2^1(3)} + \dfrac{(x-5)^2}{2^2(4)} - \dfrac{(x-5)^3}{2^3(5)}$

   (b) 0.371

CALCULUS BC
WORKSHEET ON RADIUS AND INTERVAL OF CONVERGENCE

Work the following on **notebook paper**.

Find the radius and interval of convergence for each of the following series. All work must be shown.
**(Note: Every time you are asked to find the interval of convergence, you must check to see if the endpoints are included in the interval.)**

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (x-3)^n}{n}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n!}$

3. $\displaystyle\sum_{n=1}^{\infty} n!(x+2)^n$

4. $\displaystyle\sum_{n=0}^{\infty} \frac{(x+4)^n}{3^n (n+1)}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (x-5)^n}{n^2}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-3)^n}{n5^n}$

Answers to Worksheet on Radius and Interval of Convergence

1. Radius = 1, Interval: $2 < x \le 4$

2. Radius $= \infty$, Interval: $-\infty < x < \infty$

3. Radius = 0, Interval: $\{-2\}$

4. Radius = 3, Interval: $-7 \le x < -1$

5. Radius = 1, Interval: $4 \le x \le 6$

6. Radius $= \infty$, Interval: $-\infty < x < \infty$

7. Radius $= \dfrac{5}{2}$, Interval: $-1 \le x < 4$

CALCULUS BC
WORKSHEET 1 ON POWER SERIES

Work these on **notebook paper**, except for problem 1.

1. Derive the Taylor series formula for $f(x)$ by filling in the blanks below.

   Let $f(x)=a_0+a_1(x-c)+a_2(x-c)^2+a_3(x-c)^3+a_4(x-c)^4+a_5(x-c)^5+...+a_n(x-c)^n+...$

   Find $f(c)$ and solve for $a_0$.

   $f(c)=$__________ so $a_0=$__________

   Now differentiate $f(x)$ to find $f'(x)$, and then find $f'(c)$ and solve for $a_1$..

   $f'(x)=$

   $f'(c)=$ __________ so $a_1=$ __________

   Differentiate again, this time to find $f''(x)$, and then find $f''(c)$ and solve for $a_2$.

   $f''(x)=$

   $f''(c)=$ __________ so $a_2=$__________

   Now find $f'''(x)$, and then find $f'''(c)$ and solve for $a_3$.

   $f'''(x)=$

   $f'''(c)=$__________ so $a_3=$__________

   Do you see a pattern? $f^{(n)}(c)=$______________ so $a_n=$______________

   Now substitute your results into

   $f(x)=a_0+a_1(x-c)+a_2(x-c)^2+a_3(x-c)^3+a_4(x-c)^4+a_5(x-c)^5+...+a_n(x-c)^n+...$

   $f(x)=$____$+$\_\_\_\_$(x-c)+$\_\_\_\_\_\_\_$(x-c)^2+$\_\_\_\_\_\_\_$(x-c)^3+...+$\_\_\_\_\_\_\_$(x-c)^n+...$

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On problem 2, find a Taylor series for $f(x)$ centered at the given value of $c$. Give the first four nonzero terms and the general term for the series.

2. $f(x)=e^{2x}$, $c=3$

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On problem 3 - 4, find a Taylor series for $f(x)$ centered at the given value of $c$. Give the first four nonzero terms. (You do not need to give the general term.)

3. $f(x)=\sin\left(2x+\dfrac{\pi}{3}\right)$, $c=0$

4. $f(x)=\cos x$, $c=\dfrac{2\pi}{3}$

**TURN**->>>

On problems 5 – 8, find a Maclaurin series for $f(x)$. Give the first four nonzero terms and the general term for each series.

5. $f(x)=\sin\left(x^3\right)$

6. $f(x)=\dfrac{\cos(3x)}{x}$

7. $f(x)=x^2e^{-x}$

8. $f(x)=\sin^2 x$ $\left(\text{Hint: Use the fact that } \sin^2 x=\dfrac{1-\cos(2x)}{2}.\right)$

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Answers to Worksheet 1 on Power Series

1. $a_0=f(c),\ a_1=f'(c),\ a_2=\dfrac{f''(c)}{2!},\ a_c=\dfrac{f'''(c)}{3!},\ a_n=\dfrac{f^{(n)}(c)}{n!}$

$$f(x)=f(c)+f'(c)(x-c)+\frac{f''(c)(x-c)^2}{2!}+\frac{f'''(c)(x-c)^3}{3!}+\ldots+\frac{f^{(n)}(c)(x-c)^n}{n!}+\ldots$$

2. $e^6+2e^6(x-3)+\dfrac{4e^6(x-3)^2}{2!}+\dfrac{8e^6(x-3)^3}{3!}+\ldots+\dfrac{2^n e^6(x-3)^n}{n!}+\ldots$

3. $\dfrac{\sqrt{3}}{2}+x-\dfrac{2\sqrt{3}x^2}{2!}-\dfrac{4x^3}{3!}+\ldots$

4. $-\dfrac{1}{2}-\dfrac{\sqrt{3}}{2}\left(x-\dfrac{2\pi}{3}\right)+\dfrac{\left(x-\dfrac{2\pi}{3}\right)^2}{2\cdot 2!}+\dfrac{\sqrt{3}\left(x-\dfrac{2\pi}{3}\right)^3}{2\cdot 3!}+\ldots$

5. $x^3-\dfrac{x^9}{3!}+\dfrac{x^{15}}{5!}-\dfrac{x^{21}}{7!}+\ldots+\dfrac{(-1)^n x^{6n+3}}{(2n+1)!}+\ldots$

6. $\dfrac{1}{x}-\dfrac{9x}{2!}+\dfrac{81x^3}{4!}-\dfrac{729x^5}{6!}+\ldots+\dfrac{(-1)^n 3^{2n} x^{2n-1}}{(2n)!}+\ldots$ where $x\neq 0$

7. $x^2-x^3+\dfrac{x^4}{2!}-\dfrac{x^5}{3!}+\ldots+\dfrac{(-1)^n x^{n+2}}{n!}+\ldots$

8. $\dfrac{2x^2}{2!}-\dfrac{8x^4}{4!}+\dfrac{32x^6}{6!}-\dfrac{128x^8}{8!}+\ldots+\dfrac{(-1)^{n+1} 2^{2n-1} x^{2n}}{(2n)!}+\ldots$

CALCULUS BC
WORKSHEET 2 ON POWER SERIES

Work the following on **notebook paper**. Do **not** use your calculator. Show all work.

1. (a) Find a Maclaurin series for $f(x) = \cos x$. Give the first four nonzero terms and the general term.

   (b) Use your answer to (a) to find $\lim_{x \to 0} \frac{\cos x - 1}{x^2}$.

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2. (a) Find a Maclaurin series for $f(x) = \frac{1}{1-2x}$. Give the first four nonzero terms and the general term.

   (b) Use your answer to (a) to find $\lim_{x \to 0} \frac{f(x) - 1}{x}$.

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3. (a) Find a Maclaurin series for $f(x) = \sin x$. Give the first four nonzero terms and the general term.

   (b) Use your answer to (a) to approximate the value of $\int_0^1 \frac{\sin t}{t}\,dt$ so that the error in your approximation is less than $\frac{1}{500}$. Justify your answer.

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On problems 4 - 5, find a series for the given function. Give the first four nonzero terms and the general term for the series.

4. $f(x) = e^{(x+2)}$ centered at $x = 0$

5. $g(x) = e^{(x+2)}$ centered at $x = -2$

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6. (a) Let $f(x) = \sin(x^2)$. Write the first four nonzero terms of the Taylor series for $\sin(x^2)$ about $x = 0$.

   (b) Let $g(x) = \cos(x)$. Write the first four nonzero terms of the Taylor series for $\cos(x)$ about $x = 0$.

   (c) Let $h(x) = \sin(x^2) + \cos(x)$. Write the first four nonzero terms of the Taylor series for $h$ about $x = 0$.

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7. (a) Let $f(x) = \sin(x^2)$. Write the first four nonzero terms and the general term of the Taylor series for $\sin(x^2)$ about $x = 0$.

   (b) Let $g'(x) = \sin(x^2)$. Given that $g(0) = 1$, write the first five nonzero terms and the general term of the Taylor series for $g(x)$ about $x = 0$.

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8. (1990 BC 5) Let $f$ be the function defined by $f(x) = \frac{1}{x-1}$.

   (a) Write the first four terms and the general term of the Taylor series expansion of $f(x)$ about $x = 2$.

   (b) Use the result from part (a) to find the first four terms and the general term of the series expansion about $x = 2$ for $\ln|x-1|$.

   (c) Use the series in part (b) to find an approximation for $\ln\frac{3}{2}$ so that the error in your approximation is less than $\frac{1}{20}$. How many terms were needed? Justify your answer.

Answers to Worksheet 2 on Power Series

1. (a) $1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots+\frac{(-1)^n x^{2n}}{(2n)!}+\ldots$ (b) $-\frac{1}{2}$

2. (a) $1+2x+4x^2+8x^3+\ldots+(2x)^n+\ldots$ (b) 2

3. (a) $x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots+\frac{(-1)^n x^{2n+1}}{(2n+1)!}+\ldots$

(b) $\frac{17}{18}$. Since the terms of the series are alternating, decreasing in magnitude, and having a limit of 0 and the approximation is made by using the first two terms, the error will be less than the absolute value of the third term, so $|\text{Error}|<\frac{1}{600}<\frac{1}{500}$.

4. $e^2+e^2x+\frac{e^2x^2}{2!}+\frac{e^2x^3}{3!}+\ldots+\frac{e^2x^n}{n!}+\ldots$

5. $1+(x+2)+\frac{(x+2)^2}{2!}+\frac{(x+2)^3}{3!}+\ldots+\frac{(x+2)^n}{n!}+\ldots$

6. (a) $x^2-\frac{x^6}{3!}+\frac{x^{10}}{5!}-\frac{x^{14}}{7!}+\ldots$ (b) $1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots$

(c) $1+\left(1-\frac{1}{2!}\right)x^2+\frac{x^4}{4!}-\left(\frac{1}{3!}+\frac{1}{6!}\right)x^6+\ldots=1+\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{121x^6}{6!}+\ldots$

7. (a) $x^2-\frac{x^6}{3!}+\frac{x^{10}}{5!}-\frac{x^{14}}{7!}+\ldots+\frac{(-1)^n x^{4n+2}}{(2n+1)!}+\ldots$

(b) $1+\frac{x^3}{3}-\frac{x^7}{7\cdot 3!}+\frac{x^{11}}{11\cdot 5!}-\frac{x^{15}}{15\cdot 7!}+\ldots+\frac{(-1)^n x^{4n+3}}{(4n+3)(2n+1)!}+\ldots$

8. (a) $1-(x-2)+(x-2)^2-(x-2)^3+\ldots+(-1)^n(x-2)^n+\ldots$

(b) $(x-2)-\frac{(x-2)^2}{2}+\frac{(x-2)^3}{3}-\frac{(x-2)^4}{4}+\ldots+\frac{(-1)^n(x-2)^{n+1}}{n+1}+\ldots$

(c) $\ln 2\approx\frac{3}{8}$. Two terms are needed. Since the terms of the series are alternating, decreasing in magnitude, and having a limit of 0 and the approximation is made by using the first two terms, the error will be less than the absolute value of the third term, so $|\text{Error}|<\frac{1}{24}<\frac{1}{20}=0.05$.

CALCULUS BC
WORKSHEET 3 ON POWER SERIES

Work the following on notebook paper. **No calculator** except on 6(c).
On problems 1 – 3, find a power series for the given function, centered at the given value of $c$, and find its interval of convergence. Give the first four nonzero terms and the general term of the power series.

1. $f(x)=\dfrac{1}{1+x^2},\ c=0$ 2. $f(x)=\dfrac{6x}{x+2},\ c=0$ 3. $f(x)=\dfrac{24}{4-x},\ c=1$

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4. (a) Find the first four nonzero terms of the power series for $f(x)=\sin x$, centered at $x=\dfrac{\pi}{6}$.

(You do not need to write the general term.)

(b) Find the first four nonzero terms of the power series for $g(x)=\sin\left(x-\dfrac{\pi}{6}\right)$, centered at $x=\dfrac{\pi}{6}$.

(You do not need to write the general term.)

(c) Could the answers to (a) and (b) be found by substitution, or is it necessary to find derivatives and use the Taylor formula? Explain.

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5. (1986 BC 5)

(a) Find the first four nonzero terms of the Taylor series about $x=0$ for $f(x)=\sqrt{1+x}$.

(b) Use the results found in part (a) to find the first four nonzero terms in the Taylor series about $x=0$ for $g(x)=\sqrt{1+x^3}$.

(c) Find the first four nonzero terms in the Taylor series expansion about $x=0$ for the function $h$ such that $h'(x)=\sqrt{1+x^3}$ and $h(0)=4$.

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6. (1994 BC 5)

Let $f$ be the function give by $f(x)=e^{-2x^2}$.

(a) Find the first four nonzero terms and the general term of the power series for $f(x)$ about $x=0$.

(b) Find the interval of convergence of the power series for $f(x)$ about $x=0$. Show the analysis that leads to your conclusion.

(c) Let $g$ be the function given by the sum of the first four nonzero terms of the power series for $f(x)$ about $x=0$. Show that $|f(x)-g(x)|<0.02$ for $-0.6\le x\le 0.6$.

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7. (Modification of 1996 BC 2)

The Maclaurin series for $f(x)$ is given by $1+\dfrac{x}{2!}+\dfrac{x^2}{3!}+\dfrac{x^3}{4!}+\ldots+\dfrac{x^n}{(n+1)!}+\ldots$

(a) Find $f'(0)$, $f''(0)$, $f'''(0)$, and $f^{(17)}(0)$.

(b) For what values of $x$ does the given series converge? Show your reasoning.

(c) Let $g(x)=xf(x)$. Write the Maclaurin series for $g(x)$, showing the first three nonzero terms and the general term.

(d) Write $g(x)$ in terms of a familiar function without using series. Then, write $f(x)$ in terms of the same familiar function.

**Answers to Worksheet 3 on Power Series**

1. $1-x^2+x^4-x^6+...+(-1)^n x^{2n}+...$ Int. of conv.: $-1<x<1$

2. $3x-\frac{3x^2}{2}+\frac{3x^3}{4}-\frac{3x^4}{8}+...+\frac{(-1)^n 3x^{n+1}}{2^n}+...$ Int. of conv.: $-2<x<2$

3. $8+\frac{8(x-1)}{3}+\frac{8(x-1)^2}{9}+\frac{8(x-1)^3}{27}+...+\frac{8(x-1)^n}{3^n}+...$ Int. of conv.: $-2<x<4$

4. (a) $\frac{1}{2}+\frac{\sqrt{3}\left(x-\frac{\pi}{6}\right)}{2}-\frac{\left(x-\frac{\pi}{6}\right)^2}{2\cdot 2!}-\frac{\sqrt{3}\left(x-\frac{\pi}{6}\right)^3}{2\cdot 3!}+...$

(b) $\left(x-\frac{\pi}{6}\right)-\frac{\left(x-\frac{\pi}{6}\right)^3}{3!}+\frac{\left(x-\frac{\pi}{6}\right)^5}{5!}-\frac{\left(x-\frac{\pi}{6}\right)^7}{7!}+...$

(c) The answer to (a) must be found by taking derivatives and using the Taylor formula. Since none of the derivatives of the function on (a) give a value of 0 when evaluated at $\frac{\pi}{6}$, substituting into the Maclaurin series for $\sin x$ will not give the correct series. The answer to (b) can be found by substituting into the Maclaurin series for $\sin x$ because all of the derivatives of $g(x)$ when evaluated at $\frac{\pi}{6}$ will give the same values that the derivatives of $\sin x$ gives when evaluated at 0.

5. 1986 BC 5

(a) $1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3-...$

(b) $1+\frac{1}{2}x^3-\frac{1}{8}x^6+\frac{1}{16}x^9-...$

(c) $4+x+\frac{1}{8}x^4-\frac{1}{56}x^7+\frac{1}{160}x^{10}-...$

6. 1994 BC 4

(a) $1-2x^2+\frac{(2x^2)^2}{2!}-\frac{(2x^2)^3}{3!}+...+\frac{(-1)^n(2x^2)^n}{n!}+...$

(b) Converges for all $x$, $-\infty<x<\infty$

(c) Since $f$ is an alternating series whose terms decrease in magnitude and have a limit of 0, $|\text{Error}|<|\text{5th term}|$ so $|f(x)-g(x)|<0.0112<0.02$ by the Alternating Series Remainder.

7. Modification of 1996 BC 2

(a) $f'(0)=\frac{1!}{2!}$ or $\frac{1}{2}$; $f''(0)=\frac{2!}{3!}$ or $\frac{1}{3}$; $f'''(0)=\frac{3!}{4!}$ or $\frac{1}{4}$; $f^{(17)}(0)=\frac{17!}{18!}$ or $\frac{1}{18}$

(b) Converges for all $x$, $-\infty<x<\infty$

(c) $x+\frac{x^2}{2!}+\frac{x^3}{3!}+...+\frac{x^n}{n!}+...$ \

(d) $g(x)=e^x-1$

$$f(x)=\begin{cases}\frac{e^x-1}{x} & \text{if } x\neq 0\\ 1 & \text{if } x=0\end{cases}$$

CALCULUS BC
WORKSHEET 4 ON POWER SERIES

Work the following on notebook paper. Do not use your calculator.
Find the sum of each of the following convergent series.

1. $1+\frac{2}{1!}+\frac{4}{2!}+\frac{8}{3!}+\ldots+\frac{2^n}{n!}+\ldots$

2. $1-\frac{1}{3!}+\frac{1}{5!}-\frac{1}{7!}+\ldots+\frac{(-1)^n}{(2n+1)!}+\ldots$

3. $1+\frac{1}{4}+\left(\frac{1}{4}\right)^2+\left(\frac{1}{4}\right)^3+\ldots+\left(\frac{1}{4}\right)^n+\ldots$

4. $1-\frac{100}{2!}+\frac{10,000}{4!}-\ldots+\frac{(-1)^n\cdot 10^{2n}}{(2n)!}+\ldots$

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5. Let $f(x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots+\frac{(-1)^{n+1}x^n}{n}+\ldots$.

(a) For what values of $x$ does the series for $f(x)$ converge?

(b) Find the first four nonzero terms and the general term of the power series for $g(x)=f'(x)$.

(c) Use the series found in part (b) to find the value of $g\left(\frac{1}{5}\right)$. Show the steps that lead to your answer.

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6. The function $f$ is defined by $f(x)=\frac{1}{1-x^2}$.

(a) Write the Maclaurin series for $f$. Give the first four nonzero terms and the general term. For what values of $x$ does the series converge?

(b) Find the first three nonzero terms and the general term for the Maclaurin series for $f'(x)$.

(c) Use your results from part (b) to find the sum of the infinite series $\frac{2}{3}+\frac{4}{27}+\frac{6}{243}+\ldots+(2n)\left(\frac{1}{3}\right)^{2n-1}+\ldots$ if possible. Show the steps that lead to your answer.

(d) Use your results from part (b) to find the sum of the infinite series $\frac{8}{3}+\frac{256}{27}+\frac{6144}{243}+\ldots+(2n)\left(\frac{4}{3}\right)^{2n-1}+\ldots$ if possible. If it isn't possible, explain.

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7. Let $f$ be the function defined by $f(x)=\frac{1}{1+x^2}$.

(a) Write the Maclaurin series for $f$. Give the first four nonzero terms and the general term. For what values of $x$ does the series converge?

(b) Use your answer to (a) to find the first four nonzero terms and the general term of the Maclaurin series for $\int_0^x f(t)\,dt$. For what values of $x$ does this series converge? (Remember that you are always supposed to check to see if the endpoints are included in your interval. This is understood when you are asked to "find the values of $x$ for which this series converges.")

(c) Use your answer to (b) to find the value of $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots+\frac{(-1)^n}{2n+1}+\ldots$. Show the steps that lead to your answer.

**TURN->>>**

8. The Maclaurin series for $f$ is given by $1 + 7x + \frac{7^2x^2}{2!} + \frac{7^3x^3}{3!} + ... + \frac{7^nx^n}{n!} + ...$

(a) For what values of $x$ does the series for $f$ converge?

(b) Find the first three nonzero terms and the general term for the Maclaurin series for $f'(x)$.

(c) Use your answer to (b) to find the value of $f'(1)$. Show the steps that lead to your answer.

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**Answers to Worksheet 4 on Power Series**

1. $e^2$     2. sin 1     3. $\frac{4}{3}$     4. cos 10

5. (a) Converges for $-1 < x \le 1$.

(b) $g(x) = 1 - x + x^2 - x^3 + ... + (-1)^{n+1}x^{n-1} + ...$

(c) $g\left(\frac{1}{5}\right) = 1 - \frac{1}{5} + \frac{1}{24} - \frac{1}{125} + ... + \left(-\frac{1}{5}\right)^n + ...$, the sum is $\frac{1}{1-\left(-\frac{1}{5}\right)} = \frac{5}{6}$.

6. Modification of 2006, Form B, BC 6

(a) $1 + x^2 + x^4 + x^6 + ... + x^{2n}...$ . Converges for $-1 < x < 1$.

(b) $2x + 4x^3 + 6x^5 + 8x^7 + ... + 2nx^{2n-1} + ...$. Converges for $-1 < x < 1$.

(c) The series is $f'\left(\frac{1}{3}\right)$. $f'(x)\Big|_{x=\frac{1}{3}} = \frac{2x}{(1-x^2)^2}\Big|_{x=\frac{1}{3}} = \frac{27}{32}$ or 0.844

(d) Since the limit of the terms is not zero, the series diverges so it is not possible to find the sum. (OR since this is the series from (b) with $x$ replaced by $\frac{4}{3}$, and $\frac{4}{3}$ lies outside the interval of convergence, it is not possible to find the sum.)

7. (a) $1 - x^2 + x^4 - x^6 + ... + (-1)^n x^{2n} + ...$ . Converges for $-1 < x < 1$.

(b) $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ... + (-1)^n \frac{x^{2n+1}}{2n+1} + ...$, Converges for $-1 \le x \le 1$.

(c) This is $g(1)$. $g(x) = \int_0^x \frac{1}{1+t^2}\,dt = \arctan t\Big|_0^x = \arctan x$ so $g(1) = \arctan 1 = \frac{\pi}{4}$.

8. Modification of 1983, BC 5

(a) Converges for all real numbers

(b) $7 + 7^2x + \frac{7^3x^2}{2!} + ... + \frac{7^nx^{n-1}}{(n-1)!} + ...$

(c) 
$$\begin{aligned} f'(1) &= 7 + 7^2 + \frac{7^3}{2!} + ... + \frac{7^n}{(n-1)!} + ... \\ &= 7\left(1 + 7 + \frac{7^2}{2!} + ... + \frac{7^{n-1}}{(n-1)!} + ...\right) \\ &= 7e^7 \end{aligned}$$

CALCULUS BC
WORKSHEET ON POWER SERIES AND LAGRANGE ERROR BOUND

Work the following on **notebook paper**. Use your calculator on problem 1 only.

1. Let $f$ be a function that has derivatives of all orders for all real numbers $x$ Assume that

$$f(5)=6,\ f'(5)=8,\ f''(5)=30,\ f'''(5)=48, \text{ and } \left|f^{(4)}(x)\right| \le 75$$

for all $x$ in the interval $[5, 5.2]$.

(a) Find the third-degree Taylor polynomial about $x=5$ for $f(x)$.

(b) Use your answer to part (a) to estimate the value of $f(5.2)$. What is the maximum possible error in making this estimate? Give three decimal places.

(c) Use your answer to (b) to find an interval $[a, b]$ such that $a \le f(5.2) \le b$. Give three decimal places.

(d) Could $f(5.2)$ equal 8.254? Show why or why not.

2. Let $f$ be the function given by $f(x)=\cos\left(2x+\frac{\pi}{6}\right)$ and let $P(x)$ be the third-degree Taylor polynomial for $f$ about $x=0$.

(a) Find $P(x)$.

(b) Use the Lagrange error bound to show that $\left|f\left(\frac{1}{10}\right)-P\left(\frac{1}{10}\right)\right| < \frac{1}{12,000}$.

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3. Find the first four nonzero terms of the power series for $f(x)=\sin x$ centered at $x=\frac{3\pi}{4}$.

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Find the first four nonzero terms and the general term for the Maclaurin series for each of the following, and find the interval of convergence for each series.

4. $f(x)=x\cos\left(x^3\right)$

5. $g(x)=\dfrac{x^2}{1+x}$

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Find the radius and interval of convergence for:

6. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n(x-2)^n}{3^n n^2}$

7. $\displaystyle\sum_{n=1}^{\infty}(2n)!(x-5)^n$

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Multiple Choice. Show your supporting work.

8. The coefficient of $x^6$ in the Taylor series expansion about $x=0$ for $f(x)=\sin\left(x^2\right)$ is

(A) $-\frac{1}{6}$ (B) 0 (C) $\frac{1}{120}$ (D) $\frac{1}{6}$ (E) 1

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9. If $f$ is a function such that $f'(x)=\sin\left(x^2\right)$, then the coefficient of $x^7$ in the Taylor series for $f(x)$ about $x=0$ is

(A) $\frac{1}{7!}$ (B) $\frac{1}{7}$ (C) 0 (D) $-\frac{1}{42}$ (E) $-\frac{1}{7!}$

Answers to Worksheet on Power Series and Lagrange Error Bound

1. (a) $6+8(x-5)+15(x-5)^2+8(x-5)^3$

   (b) $f(5.2)\approx P_3(5.2)=8.264$

   $|R_3(5.2)|\le 0.005$

   (c) $8.259\le f(5.2)\le 8.269$

   (d) No, $f(5.2)$ can't equal 8.254 because 8.254 does not lie in the interval found in part (c).

2. (a) $\dfrac{\sqrt{3}}{2}-x-\dfrac{2\sqrt{3}}{2!}x^2+\dfrac{4}{3!}x^3$

   (b) $\left|R_3\left(\dfrac{1}{10}\right)\right|\le\left|\dfrac{16\left(\frac{1}{10}\right)^4}{4!}\right|=\dfrac{2^4\left(\frac{1}{2^4\cdot 5^4}\right)}{4!}=\dfrac{1}{5^4\cdot 4!}=\dfrac{1}{625\cdot 24}=\dfrac{1}{15{,}000}<\dfrac{1}{12{,}000}$

3. $\dfrac{\sqrt{2}}{2}-\dfrac{\sqrt{2}}{2}\left(x-\dfrac{3\pi}{4}\right)-\dfrac{\sqrt{2}}{2\cdot 2!}\left(x-\dfrac{3\pi}{4}\right)^2+\dfrac{\sqrt{2}}{2\cdot 3!}\left(x-\dfrac{3\pi}{4}\right)^3+\ldots$

4. $x-\dfrac{x^7}{2!}+\dfrac{x^{13}}{4!}-\dfrac{x^{19}}{6!}+\ldots+(-1)^n\dfrac{x^{6n+1}}{(2n)!}+\ldots.$ Converges for all real numbers.

5. $x^2-x^3+x^4-x^5+\ldots+(-1)^n x^{n+2}+\ldots.$ Converges for $-1<x<1$.

6. Radius = 3; interval: $-1\le x\le 5$

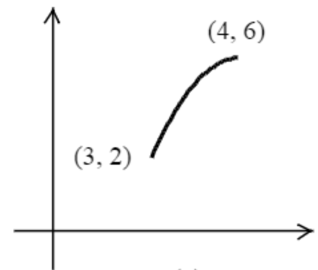
7. Converges only if $x = 5$. Radius = 0.

8. A

9. D

CALCULUS BC
WORKSHEET ON SERIES AND ERROR

1. (Calc) Let $f$ be a function that has derivatives of all orders. Assume $f(3)=1$, $f'(3)=\frac{1}{2}$, $f''(3)=-\frac{1}{4}$, $f'''(3)=\frac{3}{8}$, and the graph of $f^{(4)}(x)$ on [3, 4] is shown on the right. The graph of $f^{(4)}(x)$ is increasing on [3, 4].



Graph of $f^{(4)}(x)$

(a) Find the third-degree Taylor polynomial about $x=3$ for the function $f$.
(b) Use your answer to part (a) to estimate the value of $f(3.7)$.
(c) Use information from the graph of $y=f^{(4)}(x)$ to show that $|f(3.7)-P(3.7)|<0.08$.
(d) Could $f(3.7)$ equal 1.283? Show why or why not.

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2. (Calc) Let $f$ be the function defined by $f(x)=\sqrt{x}$.
(a) Find the second-degree Taylor polynomial about $x=4$ for the function $f$.
(b) Use your answer to part (a) to estimate the value of $f(5.1)$.
(c) Use the Lagrange error bound to find a bound on the error for the approximation in part (b).
(d) Find the value of $|f(5.1)-P_2(5.1)|$.

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3. (Calc) Find the maximum error incurred by approximating the sum of the series $1-\frac{1}{2!}+\frac{2}{3!}-\frac{3}{4!}+\frac{4}{5!}-\ldots+(-1)^{n+1}\left(\frac{n-1}{n!}\right)+\ldots$ by the sum of the first five terms. Justify your answer.

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4. Let $f$ be the function given by $f(x)=\cos\left(3x+\frac{\pi}{6}\right)$ and let $P(x)$ be the fourth-degree Taylor polynomial for $f$ about $x=0$.
(a) Find $P(x)$. (b) Use the Lagrange error bound to show that $\left|f\left(\frac{1}{6}\right)-P\left(\frac{1}{6}\right)\right|<\frac{1}{3000}$.

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5. Use series to find an estimate for $\int_0^1 e^{-x^2}\,dx$ so that the error is less than $\frac{1}{200}$. Justify your answer.

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6. (Calc) Suppose a function $f$ is approximated with a fourth-degree Taylor polynomial about $x=1$. If the maximum value of the fifth derivative between $x=1$ and $x=3$ is 0.01, that is, $\left|f^{(5)}(x)\right|<0.01$, find the maximum error incurred using this approximation to compute $f(3)$.

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7. The Taylor series about $x=5$ for a certain function $f$ converges to $f(x)$ for all $x$ in the interval of convergence. The nth derivative of $f$ at $x=5$ is given by $f^{(n)}(5)=\frac{(-1)^n n!}{2^n(n+2)}$ and $f(5)=\frac{1}{2}$.
(a) Write the third-degree Taylor polynomial for $f$ about $x=5$.
(b) Show that the third-degree Taylor polynomial for $f$ about $x=5$ approximates $f(6)$ with an error less than 0.02.

**Answers to Worksheet on Series and Error**

1. (a) $1+\dfrac{x-3}{2}-\dfrac{(x-3)^2}{4\cdot 2!}+\dfrac{3(x-3)^2}{8\cdot 3!}$

   (b) 1.310

   (c) Since $f^{(4)}(x)$ is increasing on [3, 4], $f^{(4)}(x)<6$ on [3, 3.7] so

   $$|\text{Error}|<\left|\frac{6(3.7-3)^4}{4!}\right|=0.060<0.08.$$

   (d) Yes, $1.250\le f(3.7)\le 1.370$ so $f(3.7)$ could equal 1.283.

2. (a) $2+\dfrac{x-4}{4}-\dfrac{(x-4)^2}{32\cdot 2!}$

   (b) 2.256

   (c) The maximum value of the third derivative $f'''(x)=\dfrac{3}{8x^{5/2}}$ on [4, 5.1] is

   $$f'''(4)=\frac{3}{8(4)^{5/2}}=\frac{3}{256} \text{ so } |\text{Error}|\le\left|\frac{\frac{3}{256}(5.1-4)^3}{3!}\right|=0.003.$$

   (d) 0.002

3. The series has terms that are alternating in sign, decreasing in magnitude, and having a limit of 0 so the error is less than the absolute value of the first truncated term by the Alternating Series Remainder.

   $|\text{Error}|<|\text{6th term}|$ so $|\text{Error}|<\dfrac{5}{6!}$ or 0.012.

4. (a) $P(x)=\dfrac{\sqrt{3}}{2}-\dfrac{3x}{2}-\dfrac{9\sqrt{3}x^2}{2\cdot 2!}+\dfrac{27x^3}{2\cdot 3!}+\dfrac{81\sqrt{3}x^4}{2\cdot 4!}$

   (b) $|R_4(x)|=\left|\dfrac{f^{(5)}(z)(x-0)^5}{5!}\right|\le\left|\dfrac{243x^5}{5!}\right|$ so $\left|R_4\left(\dfrac{1}{6}\right)\right|\le\left(\dfrac{243}{5!}\right)\cdot\left(\dfrac{1}{6}\right)^5=\dfrac{1}{5!2^5}=\dfrac{1}{(120)(32)}<\dfrac{1}{3000}$

5. The series has terms that are alternating in sign, decreasing in magnitude, and having a limit of 0 so the error is less than the first truncated term by the Alternating Series Remainder.

   $\int_0^1 e^{-x^2}\,dx\approx 1-\dfrac{1}{3}+\dfrac{1}{10}-\dfrac{1}{42}=\dfrac{43}{105}$. $|\text{Error}|<\dfrac{1}{216}<\dfrac{1}{200}$.

6. 0.003

7. (a) $\dfrac{1}{2}-\dfrac{x-5}{2^1(3)}+\dfrac{(x-5)^2}{2^2(4)}-\dfrac{(x-5)^3}{2^3(5)}$

   Since the series has terms that are alternating, decreasing in magnitude, and having a limit of 0. the error involved in approximating $f(6)$ with the third-degree Taylor polynomial is less than the absolute value of the fourth-degree term so

   $|\text{Error}|<\dfrac{(6-5)^4}{2^4(6)}=\dfrac{1}{96}<\dfrac{1}{50}$ by the Alternating Series Remainder.

CALCULUS BC
REVIEW SHEET 1 ON SERIES

Work the following on **notebook paper**. Use your calculator only on problems 7 and 9(a) and (b).
On problems 1 and 2, find the radius of convergence and the interval of convergence.

1. $\displaystyle\sum_{n=1}^{\infty}\frac{(x-5)^n}{n3^n}$ 2. $\displaystyle\sum_{n=1}^{\infty}\frac{(x-2)^n}{n!}$

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3. Find the sum of $1-\dfrac{1}{3!}+\dfrac{1}{5!}-\dfrac{1}{7!}+\ldots+\dfrac{(-1)^n}{(2n+1)!}+\ldots$

4. What is the value of $\displaystyle\sum_{n=0}^{\infty}\frac{3^{n+1}}{4^n}$?

5. A function $f$ has a Maclaurin series given by $x^2-\dfrac{x^4}{3!}+\dfrac{x^6}{5!}-\dfrac{x^8}{7!}+\ldots$ What is $f(x)$?

6. If $f$ is a function such that $f'(x)=\cos\left(x^3\right)$, find the coefficient of the $x^7$ term in the Taylor polynomial for $f(x)$ about $x=0$.

7. The graph of the function represented by the Maclaurin series
$1-x+\dfrac{x^2}{2!}-\dfrac{x^3}{3!}+\ldots+\dfrac{(-1)^n x^n}{n!}+\ldots$ intersects the graph of $y=x^3$ at $x=$?

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8. A function $f$ is defined by $f(x)=\dfrac{1}{4}+\dfrac{2}{4^2}x+\dfrac{3}{4^3}x^2+\ldots+\dfrac{n+1}{4^{n+1}}x^n+\ldots$
for all $x$ in the interval of convergence of the power series.
(a) Find the interval of convergence for this power series. Show the work that leads to your answer.
(b) Find $\displaystyle\lim_{x\to 0}\frac{f(x)-\frac{1}{4}}{x}$.
(c) Write the first three nonzero terms and the general term for an infinite series that represents $\int_0^1 f(x)\,dx$.
(d) Find the sum of the series determined in part (c).

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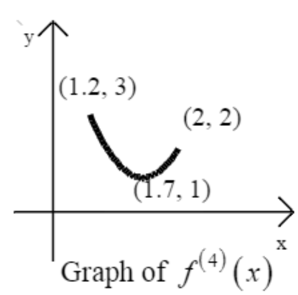
9. The function $f$ has derivatives of all orders for all real numbers $x$. Assume $f(2)=3$,
$f'(2)=-4$, $f''(2)=-1$, and $f'''(2)=5$.
(a) Write the third-degree Taylor polynomial for $f$ about $x=2$, and use it to approximate $f(1.2)$.
(b) The graph of the fourth derivative of $f$ on the interval $1.2,\ 2]$ is shown on the right. The graph of $f^{(4)}(x)$ has a horizontal tangent at $x=1.7$. Use the Lagrange error bound on the approximation of $f(1.2)$ to explain why $f(1.2)\neq 5.3$.



Graph of $f^{(4)}(x)$

(c) Write the fourth-degree Taylor polynomial, $P(x)$, for $g(x)=f\left(x^2+2\right)$ about $x=0$. Use $P$ to explain whether $g$ has a relative maximum, relative minimum, or neither at $x=0$.

**Answers to Review Sheet 1 on Series**

1. 3; $2 \le x < 8$
2. $\infty$; converges for all $x$
3. $\sin 1$
4. 12
5. $x \sin x$
6. $-\dfrac{1}{14}$
7. 0.773
8. (a) $-4 < x < 4$

   (b) $\dfrac{1}{8}$

   (c) $\dfrac{1}{4} + \dfrac{1}{16} + \dfrac{1}{64} + \dfrac{1}{256} + \ldots + \dfrac{1}{4^{n+1}} + \ldots$

   (d) $\dfrac{1}{3}$

9. (a) $3 - 4(x-2) - \dfrac{(x-2)^2}{2!} + \dfrac{5(x-2)^3}{3!}$; 5.453

   (b) $\left|R_3(1.2)\right| \le \dfrac{3(1.2-2)^4}{4!} = 0.0512$ so $5.402 \le f(1.2) \le 5.505$. Therefore $f(1.2) \ne 5.3$.

   (c) $3 - 4x^2 - \dfrac{x^4}{2!}$

   Since $g'(0) = 0$ and $g''(0) = -8$, $g$ must have a relative maximum at $x = 0$ by the Second Derivative Test.

CALCULUS BC
REVIEW SHEET 2 ON SERIES

Work the following on **notebook paper**. Do not use your calculator.

1. Which of the following is a term in the Taylor series about $x = 0$ for the function $f(x) = \cos(2x)$?

(A) $-\frac{1}{2}x^2$ (B) $-\frac{4}{3}x^3$ (C) $\frac{2}{3}x^4$ (D) $\frac{1}{60}x^5$ (E) $\frac{4}{45}x^6$

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2. Find the values of $x$ for which the series $\sum_{n=1}^{\infty}\frac{(x-2)^n}{n(-3)^n}$ converges.

(A) $x = 2$ (B) $-1 \le x < 5$ (C) $-1 < x \le 5$ (D) $-1 < x < 5$ (E) All real numbers

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3. The series $x + x^3 + \frac{x^5}{2!} + \frac{x^7}{3!} + ... + \frac{x^{2n+1}}{n!} + ...$ is the Maclaurin series for

(A) $x\ln\left(1+x^2\right)$ (B) $x\ln\left(1-x^2\right)$ (C) $e^{x^2}$ (D) $xe^{x^2}$ (E) $x^2e^{x^2}$

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4. The coefficient of $x^3$ in the Taylor series for $e^{2x}$ at $x = 0$ is

(A) $\frac{1}{6}$ (B) $\frac{1}{3}$ (C) $\frac{2}{3}$ (D) $\frac{4}{3}$ (E) $\frac{8}{3}$

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5. Let $f(x) = \sum_{n=1}^{\infty}(\cos x)^{3n}$. Evaluate $f\left(\frac{2\pi}{3}\right)$.

6. Find the sum of the geometric series $\frac{9}{8} - \frac{3}{4} + \frac{1}{2} - \frac{1}{3} + ...$

7. Find the Taylor polynomial of order 3 at $x = 0$ for $f(x) = \sqrt{1+x}$.

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(2003 Form B, Problem 6)

8. The function $f$ has a Taylor series about $x = 2$ that converges to $f(x)$ for all $x$ in the interval of convergence. The nth derivative of $f$ at $x = 2$ is given by $f^{(n)}(2) = \frac{(n+1)!}{3^n}$ for $n \ge 1$, and $f(2) = 1$.

(a) Write the first four terms and the general term of the Taylor series for $f$ about $x = 2$.
(b) Find the radius of convergence for the Taylor series for $f$ about $x = 2$. Show the work that leads to your answer.
(c) Let $g$ be a function satisfying $g(2) = 3$ and $g'(x) = f(x)$ for all $x$. Write the first four terms and the general term of the Taylor series for $g$ about $x = 2$.
(d) Does the Taylor series for $g$ as defined in part (c) converge at $x = -2$? Give a reason for your answer.

9. Let $f$ be the function given by $f(x)=\cos\left(3x+\frac{3\pi}{4}\right)$ and let $P(x)$ be the third-degree Taylor polynomial for $f$ about $x=0$.

(a) Find $P(x)$.

(b) Use the Lagrange error bound to show that $\left|f\left(\frac{1}{6}\right)-P\left(\frac{1}{6}\right)\right|\le\frac{1}{300}$.

(c) Let $G(x)$ be the function given by $G(x)=\int_0^x f(t)\,dt$. Write the third-degree Taylor polynomial, $T(x)$, for $G$ about $x=0$.

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10. Suppose that the function $f(x)$ is approximated near $x=5$ by a fourth-degree Taylor polynomial $P_4(x)=3+7(x-5)^2-2(x-5)^3+9(x-5)^4$.

(a) Find the value of $f(5)$, $f'(5)$, $f''(5)$, $f'''(5)$, and $f^{(4)}(5)$.

(b) Does $f$ have a local maximum, a local minimum, or neither at $x=5$? Justify your answer.

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**<u>Answers to Review Sheet 2 on Series</u>**

| | | | |
|---|---|---|---|
| 1. C | 2. C | 3. D | 4. D |
| 5. $-\frac{1}{9}$ | 6. $\frac{27}{40}$ | 7. $1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}$ | |

8. (2003 Form B, Problem 6)

(a) $1+\frac{2(x-2)}{3}+\frac{3(x-2)^2}{3^2}+\frac{4(x-2)^3}{3^3}+\ldots+\frac{(n+1)(x-2)^n}{3^n}+\ldots$

(b) Radius = 3

(c) $1+x+\frac{(x-2)^2}{3}+\frac{(x-2)^3}{9}+\frac{(x-2)^4}{27}+\ldots+\frac{(x-2)^{n+1}}{3^n}+\ldots$

or $3+(x-2)+\frac{(x-2)^2}{3}+\frac{(x-2)^3}{9}+\frac{(x-2)^4}{27}+\ldots+\frac{(x-2)^{n+1}}{3^n}+\ldots$

(d) No, $x=-2$ is outside the interval of convergence, which is $-1<x<5$.

9. (a) $P(x)=-\frac{\sqrt{2}}{2}-\frac{3\sqrt{2}x}{2}+\frac{9\sqrt{2}x^2}{2\cdot 2!}+\frac{27\sqrt{2}x^3}{2\cdot 3!}$

(b) Since $f^{(4)}(x)=81\cos\left(3x+\frac{3\pi}{4}\right)$ and $-1\le\cos\left(3x+\frac{3\pi}{4}\right)\le 1$ for all $x$ in the interval $o<x<\frac{1}{6}$,

$$|R_3(x)|=\left|\frac{f^{(4)}(z)(x-0)^4}{4!}\right|\le\left|\frac{81x^4}{4!}\right| \text{ so } \left|R_3\left(\frac{1}{6}\right)\right|\le\left|\frac{81\left(\frac{1}{6}\right)^4}{4!}\right|=\frac{3^4}{(4!)(3^4)(2^4)}=\frac{1}{(24)(16)}<\frac{1}{300}$$

(c) $-\frac{\sqrt{2}}{2}x-\frac{3\sqrt{2}}{4}x^2+\frac{3\sqrt{2}}{4}x^3$

10. (a) $f(5)=3$, $f'(5)=0$, $f''(5)=14$, $f'''(5)=-12$, $f^{(4)}(5)=216$

(b) Since $f'(5)=0$ and $f''(5)=14$ and $14>0$, $f$ must have a relative minimum at $x=5$ by the Second Derivative Test.